[redacted]

1 The expression $7\cos\theta + 4\sin\theta$ is defined for $0 \leqslant \theta \leqslant \pi$ radians.

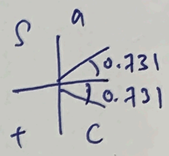
(i) Using $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, solve the equation $7\cos\theta + 4\sin\theta = 6$. [4]

$R\cos(\theta - \alpha) = R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$

$R\cos\alpha = 7$, $R\sin\alpha = 4$ } $\tan\alpha = \frac{4}{7}$

$\alpha = 0.519$



$8.062\cos(\theta - 0.519) = 6$

basic $\angle = 0.731$

$\theta - 0.519 = 0.731, \ -0.731$

$\theta = 1.25\text{ rad}, \ -0.212\text{ rad}$ (rej)

(ii) State the largest and smallest values of $80 - (7\cos\theta + 4\sin\theta)^2$ and find the corresponding values of θ. [4]

$80 - (8.062\cos(\theta - 0.519))^2$

largest value $= 80$ when $\theta = 1.0517$

smallest value $= 15$ when $\theta = 0.519$

2 (a) Solve the simultaneous equations

$$y = 2x^2 - 7,$$
$$y = 3x + 20.$$

[3]

$2x^2 - 7 = 3x + 20$

$2x^2 - 3x - 27 = 0$

$(2x - 9)(x + 3) = 0$

$x = \frac{9}{2}$ or -3

$y = \frac{67}{2}$ or 11

(b) Find the greatest value of the integer a for which $ax^2 + 5x - 2$ is negative for all x. [3]

$b^2 - 4ac < 0$

$25 - 4(a)(-2) < 0$

$8a < -25$

$a < -\frac{25}{8}$

$a = -4$

Suggested Answers

(c) Find the values of the constant c for which the line $y = 4x + c$ is a tangent to the curve $y = x^2 + cx + \frac{21}{4}$. [3]

$$x^2 + cx + \frac{21}{4} = 4x + c$$

$$x^2 + (c-4)x + \frac{21}{4} - c = 0$$

$$b^2 - 4ac = 0$$

$$(c-4)^2 - 4(1)\left(\frac{21}{4} - c\right) = 0$$

$$c^2 - 8c + 16 - 21 + 4c = 0$$

$$c^2 - 4c - 5 = 0$$

$$(c+1)(c-5) = 0$$

$$c = -1 \quad \text{or} \quad 5$$

3 **(i)** By considering the general term in the binomial expansion of $\left(\frac{3}{x^2}+x\right)^8$, explain why every term is dependent on x. [3]

$T_{n+1} = {}^8C_n \left(\frac{3}{x^2}\right)^{8-n} x^n$

$= {}^8C_n \, 3^{8-n} \, x^{-16+2n} \cdot x^n$

$= {}^8C_n \, 3^{8-n} \, x^{-16+3n}$

$-16+3n=0$

$n = \frac{16}{3} \notin \mathbb{Z}^+$

$\therefore$ every term is dependent on x.

(ii) Find the term independent of x in the expansion of $\left(\frac{3}{x^2}+x\right)^8(5-2x)$. [3]

$-16+3n=-1$

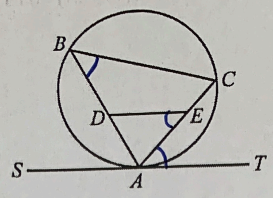
$3n = 15$

$n=5$

Term independent of x: ${}^8C_5 \, 3^{8-5} (-2)$

$= -3024$

Suggested Answers



The diagram shows a circle passing through the points A, B and C. The straight line SAT is a tangent to the circle. The points D and E lie on AB and AC respectively and are such that $BCED$ is a cyclic quadrilateral. Prove that DE is parallel to ST. [5]

$\angle CAT = \angle CBA$ (Alt. seg thm)

$\because$ BCED is cyclic quad

then $\angle DEC = 180^\circ - \angle DBC$

$= 180^\circ - \angle CBA$ ($\because \angle DBC = \angle CBA$)

thus $\angle AED = 180^\circ - \angle DEC$ ($\angle$ in str. line)

$= 180^\circ - (180^\circ - \angle CBA)$

$= \angle CBA$

$\therefore DE \parallel ST$ (Alt. int. $\angle$)

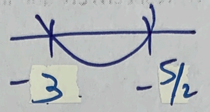
Suggested Answers

5 (a) Represent the solution set of $15(1+2x) \geqslant x(19-2x)$ on a number line. [4]

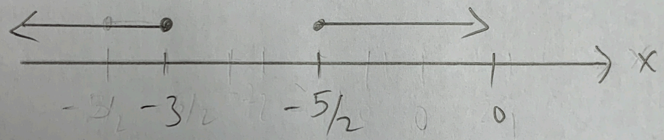
$$15+30x-19x+2x^2 \geqslant 0$$

$$2x^2+11x+15 \geqslant 0$$

$$(2x+5)(x+3) \geqslant 0$$

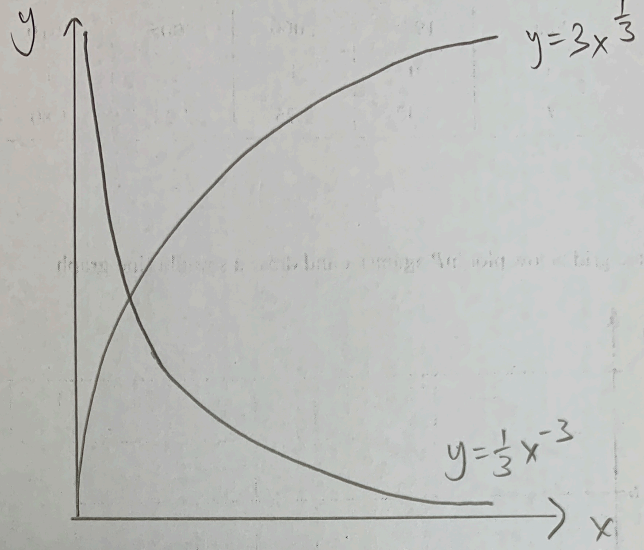


$x \leq -3$ $\quad$ $x \geqslant -\frac{5}{2}$



Suggested Answers

(b) (i) Clearly labelling each graph, sketch, on the same axes, the graphs of $y = 3x^{\frac{1}{3}}$ and $y = \frac{1}{3}x^{-3}$ for $x > 0$. [2]



(ii) Show that the x-coordinate of the point of intersection of your graphs satisfies the equation $x^{10} = \frac{1}{729}$. [2]

$$3x^{\frac{1}{3}} = \frac{1}{3x^3}$$

$$9x^{\frac{10}{3}} = 1$$

$$x^{\frac{10}{3}} = \frac{1}{9}$$

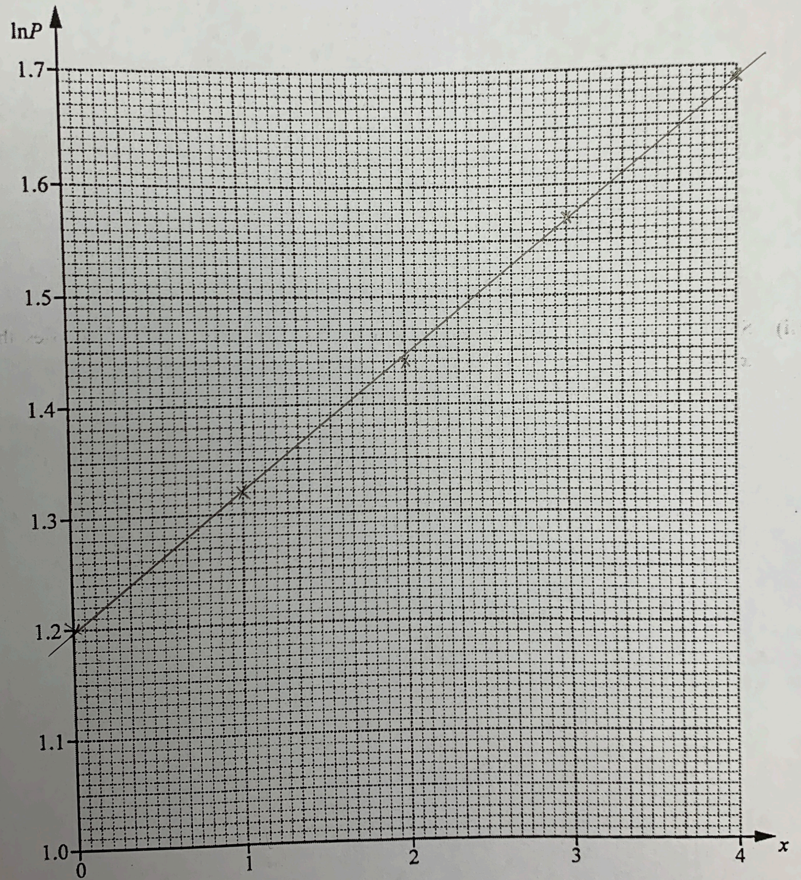
$$x^{10} = \frac{1}{9^3} = \frac{1}{729}$$

[redacted]

6 The table shows, to 3 significant figures, the population, P, in millions, of a country on January 1st at intervals of five years from 1995 to 2015. The variable x is measured in units of 5 years.

Year	1995	2000	2005	2010	2015
x	0	1	2	3	4
P	3.32	3.75	4.24	4.80	5.43
ln P	1.199	1.322	1.444	1.569	1.692

(i) On the grid below plot $\ln P$ against x and draw a straight line graph. [2]



(ii) Find the gradient of your straight line and hence express P in the form Ae^{kx}, where A and k are constants. [4]

$$m = \frac{1.692 - 1.199}{4 - 0} = 0.12325$$

$$Y = mX + c$$

$$\ln P = 0.12325x + 1.199$$

$$P = e^{0.12325x} \cdot e^{1.199}$$

$$= 3.32\, e^{0.123x} \quad (3 \text{ sf})$$

(iii) If this model for the population remains valid, find the first year of the interval in which the population first exceeds 8 million. [3]

$$8 = 3.32\, e^{0.12325x}$$

$$x = 7.135$$

# of yrs : $7.135 \times 5 = 35.65$ yrs

≈ 35 yrs & 7 mths.

(Jul 2030)

Yr	1995	2000	....	2030	2035
x	0	1	...	7	8

interval : 2030 — 2035

∴ 1st yr of interval 2030

7 (i) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{x(3x-5)^{\frac{5}{3}}\right\} = (8x-5)(3x-5)^{\frac{2}{3}}$. [5]

$$\frac{d}{dx}\, x(3x-5)^{5/3}$$

$$= (3x-5)^{5/3} + \frac{5}{3}(3x-5)^{2/3}(3)(x)$$

$$= (3x-5)(3x-5)^{2/3} + 5x(3x-5)^{2/3}$$

$$= (8x-5)(3x-5)^{2/3}$$

(ii) Hence find $\int x(3x-5)^{\frac{2}{3}}\,dx$, giving your answer in the form $k(x+1)(3x-5)^{\frac{5}{3}}+c$, where k is a constant to be found and c is a constant of integration which cannot be found. [4]

$$\int (8x-5)(3x-5)^{2/3}\,dx = x(3x-5)^{5/3}+c$$

$$8\int x(3x-5)^{2/3}\,dx - 5\int (3x-5)^{2/3}\,dx = x(3x-5)^{5/3}+c$$

$$\int x(3x-5)^{2/3}\,dx = \frac{1}{8}x(3x-5)^{5/3} + \frac{5}{8}\left[\frac{(3x-5)^{5/3}}{(5/3)(3)}\right]+c$$

$$= \frac{x}{8}(3x-5)^{5/3} + \frac{1}{8}(3x-5)^{5/3}+c$$

$$= \frac{1}{8}(x+1)(3x-5)^{5/3}+c$$

$$\therefore k = \frac{1}{8}$$

(iii) Find the value of $\int_{-1}^{\frac{5}{3}} x(3x-5)^{\frac{2}{3}}\,dx$ and explain what the result implies about the curve $y = x(3x-5)^{\frac{2}{3}}$. [2]

$$\int_{-1}^{5/3} x(3x-5)^{2/3}\,dx = \left[\frac{1}{8}(x+1)(3x-5)^{5/3}\right]_{-1}^{5/3}$$

$$= 0 - (0) = 0$$

The curve $x(3x-5)^{2/3}$ cuts the x-axis between -1 to $5/3$. Let this x-intercept be α. Then the area under the curve from -1 to α is equal to the area under the curve from α to $5/3$.

8 (a) Solve the equation $e^x(1+e^x)=\frac{3}{4}$. [3]

let $e^x = y$

$y^2+y=\frac{3}{4}$

$y^2+y-\frac{3}{4}=0$

$y=\frac{1}{2}$ or $-\frac{3}{2}$

$e^x=\frac{1}{2}$ or $e^x=-\frac{3}{2}$

$x=\ln\frac{1}{2}$ rej

$=-\ln 2$

(b) Solve the equation $1+\log_2 y+\frac{1}{\log_8 2}=\log_2(y+3)$. [4]

$1+\log_2 y+\log_2 8=\log_2(y+3)$

$4+\log_2 y=\log_2(y+3)$

$4=\log_2\left(\frac{y+3}{y}\right)$

$16=\frac{y+3}{y}$

$16y=y+3$

$15y=3 \quad \therefore y=\frac{1}{5}$

(c) In order to obtain a graphical solution of the equation $x = \ln\left\{\left(\frac{2x+7}{3}\right)^2\right\}$ a suitable straight line can be drawn on the same set of axes as the graph of $y = 3e^{\frac{x}{2}} + 4$. Make $e^{\frac{x}{2}}$ the subject of $x = \ln\left\{\left(\frac{2x+7}{3}\right)^2\right\}$ and hence find the equation of this line. [3]

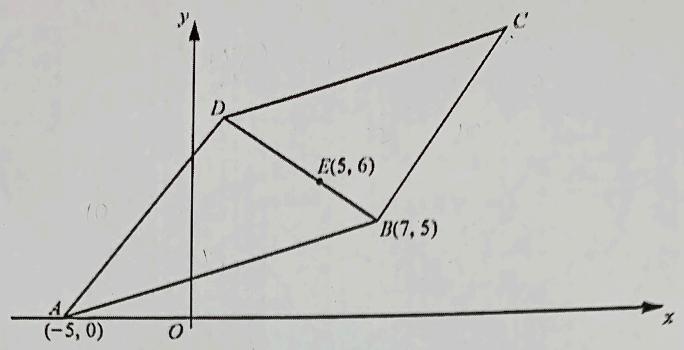
$$\frac{x}{2} = \ln \frac{2x+7}{3}$$

$$e^{x/2} = \frac{2x+7}{3}$$

Sub in $y = 3\left(\frac{2x+7}{3}\right) + 4$

$$= 2x + 11$$

9



The diagram shows a parallelogram with vertices $A(-5, 0)$, $B(7, 5)$, C and D. The point $E(5, 6)$ lies on the diagonal BD of the parallelogram. The perimeter of the parallelogram is 46 units.

(i) Show that the x-coordinate of D satisfies the equation $5x^2 + 6x = 11$. [7]

$\text{Grad}_{DB} = \frac{6-5}{5-7} = -\frac{1}{2}$

Eq^{n} of DB : $y - 6 = -\frac{1}{2}(x-5)$

$y = -\frac{1}{2}x + \frac{5}{2} + 6$

$= -\frac{1}{2}x + \frac{17}{2}$

length of AB = 13 = length of DC

∴ length of AD = 10

let coordinate D be (x, y)

$y^2 + (x+5)^2 = 100$

sub in $y = -\frac{1}{2}x + \frac{17}{2}$

$\left(\frac{-1}{2}x + \frac{17}{2}\right)^2 + (x+5)^2 = 100$

$\frac{1}{4}x^2 - \frac{17}{2}x + \frac{289}{4} + x^2 + 10x + 25 = 100$

$\frac{5}{4}x^2 + \frac{3}{2}x - \frac{11}{4} = 0$

$\Rightarrow 5x^2 + 6x = 11$ //

(ii) Determine the coordinates of D, explaining why the diagram is necessary. [3]

$5x^2+6x-11=0$

$(5x+11)(x-1)=0$

$\therefore x=-\frac{11}{5}$ or 1

Two x-coordinates are found, using the diagram, the x-coordinate of D is positive

$\therefore x=1$

$y=8$

(iii) Find the coordinates of C. [1]

By similar $\triangle$, $C(13,13)$

10 The function f is defined for $x \in \mathbb{R}$ and is such that $f''(x) = 48x^2 + 2e^{2x-1}$.

The line $x = \frac{1}{2}$ is the normal to the curve $y = f(x)$ at the point where $y = \frac{1}{4}$.

(i) Find an expression for $f'(x)$. [5]

$\therefore y = \frac{1}{4}$ is tangent $\Rightarrow$ turning pt at $x = \frac{1}{2}$

$f''(x) = 48x^2 + 2e^{2x-1}$

$f'(x) = 16x^3 + e^{2x-1} + c$

$f'(\frac{1}{2}) = 0$

$16(\frac{1}{8}) + e^0 + c = 0$

$2 + 1 + c = 0$

$c = -3$

$\therefore f'(x) = 16x^3 + e^{2x-1} - 3$

(ii) Hence find an expression for f(x). [4]

$f(x) = \frac{16}{4}x^4 + \frac{1}{2}e^{2x-1} - 3x + d$

$\because f(\frac{1}{2}) = \frac{1}{4}$

$4(\frac{1}{2})^4 + \frac{1}{2}e^0 - 3(\frac{1}{2}) + d = \frac{1}{4}$

$d = 1$

$\therefore f(x) = 4x^4 + \frac{1}{2}e^{2x-1} - 3x + 1$

(iii) Show that the equation of the tangent to the curve $y = f(x)$ at the point where the curve intersects the y-axis can be written as

$$2e(y + 3x - 1) = 2x + 1.$$ [3]

$f'(0) = e^{-1} - 3 \qquad f(0) = \frac{1}{2}e^{-1} + 1$

Eqn of tangent $y - (\frac{1}{2}e^{-1} + 1) = (\frac{1}{e} - 3)(x - 0)$

$y - \frac{1}{2e} - 1 = \frac{x}{e} - 3x$

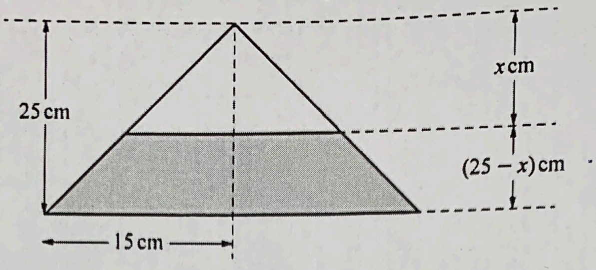
$2ey - 1 - 2e = 2x - 6ex$

$2ey + 6ex - 2e = 2x + 1$

$2e(y + 3x - 1) = 2x + 1$

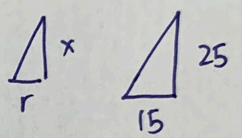
4047/02/O/N/20 [Turn over

11 [The volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.]



The diagram shows a vertical cross-section of a container in the form of a cone, height 25 cm and base radius 15 cm. The container is initially empty. At time $t = 0$ liquid is allowed to flow in through a small hole close to the vertex. After t seconds the height of liquid in the container is $(25 - x)$ cm and the volume of liquid is V cm^3.

(i) Show that $V = 3\pi\left(625 - \frac{x^3}{25}\right)$. [3]



$r = \frac{x(15)}{25} = \frac{3}{5}x$

$V = \frac{1}{3}\pi(15^2)(25) - \frac{1}{3}\pi\left(\frac{3}{5}x\right)^2(x)$

$= 1875\pi - \frac{3}{25}\pi x^3$

$= 3\pi\left(625 - \frac{x^3}{25}\right)$ //

Given that the rate of flow of the liquid is kx^2 cm^3/s, where k is a constant,

(ii) find an expression for $\frac{dt}{dx}$ in terms of π and k. [4]

$$\frac{dV}{dt} = kx^2$$

$$\frac{dV}{dx} = -\frac{9\pi x^2}{25}$$

$$\frac{dt}{dx} = \frac{dt}{dV}\left(\frac{dV}{dx}\right)$$

$$= \frac{1}{kx^2}\left(-\frac{9\pi x^2}{25}\right)$$

$$= -\frac{9\pi}{25k}$$

The liquid was allowed to flow for 72π seconds when the height of the liquid reached 12 cm.

(iii) By expressing t as a function of x, find the value of k. [4]

$$\frac{dt}{dx} = -\frac{9\pi}{25k}$$

$$t = -\frac{9\pi}{25k}x + c$$

$t = 0 \quad x = 25 \quad c = \frac{9\pi}{k}$

$$t = -\frac{9\pi}{25k}x + \frac{9\pi}{k}$$

$t = 72\pi$, $x = 25 - 12 = 13$

$$72\pi = -\frac{9\pi}{25k}(13) + \frac{9\pi}{k}$$

$$72 = \frac{108}{25k}$$

$$k = \frac{3}{50}$$ #

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