# 2019 GCE A'Level
## H2 Mathematics Paper 1 (9758/01)

## Suggested Answers

studies
where transFormAtion begins

**1** The function f is defined by $f(z) = az^3 + bz^2 + cz + d$, where $a$, $b$, $c$ and $d$ are real numbers. Given that $2 + \mathrm{i}$ and $-3$ are roots of $f(z) = 0$, find $b$, $c$ and $d$ in terms of $a$. [4]

① $\because$ $2+i$ is a root, then $2-i$ is also a root. $\because a, b, c, d \in \mathbb{R}$

$$f(z) = (z-(2+i))(z-(2-i))(z+3)(a)$$
$$= [(z-2)+i][(z-2)-i](z+3)(a)$$
$$= ((z-2)^2+1)(z+3)(a)$$
$$= (z^2-4z+5)(z+3)(a)$$
$$= (z^3+3z^2-4z^2-12z+5z+15)(a)$$
$$\therefore f(z) = az^3 - az^2 - 7az + 15a$$

Hence $b = -a$ $\quad -7a = c$ $\quad d = 15a$ #

**2** The curve $C$ has equation $y = x^3 + x - 1$.

**(i)** $C$ crosses the $x$-axis at the point with coordinates $(a, 0)$. Find the value of $a$ correct to 3 decimal places. [1]

**(ii)** You are given that $b > a$.

The region $P$ is bounded by $C$, the $x$-axis and the lines $x = -1$ and $x = 0$. The region $Q$ is bounded by $C$, the line $x = b$ and the part of the $x$-axis between $x = a$ and $x = b$. Given that the area of $Q$ is 2 times the area of $P$, find the value of $b$ correct to 3 decimal places. [4]

② (i) From G.C. $\quad a = 0.682$

(ii) $\int_{-1}^{0} |x^3+x-1| \, dx = 1.75$ from G.C.

$$\int_a^b x^3+x-1 \, dx = 3.5$$
$$\left[\frac{1}{4}x^4 + \frac{1}{2}x^2 - x\right]_a^b = 3.5$$
$$\because a = 0.682$$
$$\left(\frac{1}{4}b^4 + \frac{1}{2}b^2 - b\right) - (-0.3953) = 3.5$$
$$\Rightarrow \frac{1}{4}b^4 + \frac{1}{2}b^2 - b - 3.1047 = 0$$

From G.C. $\quad b = 1.892 \quad \because b > a$

#

**3** A function is defined as $f(x) = 2x^3 - 6x^2 + 6x - 12$.

**(i)** Show that $f(x)$ can be written in the form $p\{(x+q)^3 + r\}$, where $p$, $q$ and $r$ are constants to be found. [2]

**(ii)** Hence, or otherwise, describe a sequence of transformations that transform the graph of $y = x^3$ onto the graph of $y = f(x)$. [3]

③ (i) $f(x) = 2(x^3 - 3x^2 + 3x - 6)$

$= 2((x-1)^3 - 5)$ //

(ii) ①: Translation 1 unit in the positive x-axis direction.

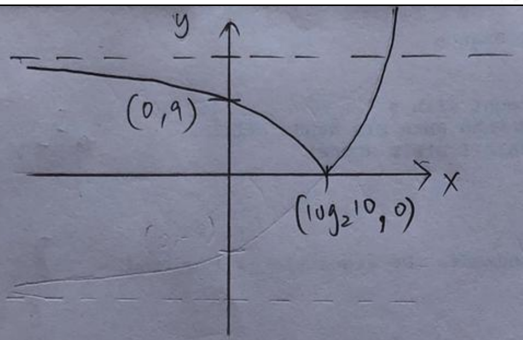
②: Translation 5 units in the negative y-axis direction.

③: Scaling 2 units parallel to the y-axis

**4** **(i)** Sketch the graph of $y = |2^x - 10|$, giving the exact values of any points where the curve meets the axes. [3]

**(ii)** Without using a calculator, and showing all your working, find the exact interval, or intervals, for which $|2^x - 10| \leqslant 6$. Give your answer in its simplest form. [3]

④ (i)




$2^x - 10 = 0$

$2^x = 10$

$x = \log_2 10$

(ii) $10 - 2^x = 6$

$2^x = 4$

$x = \log_2 4$

$= 2$

$2^x - 10 = 6$

$x = 4$

Hence, for $|2^x - 10| \leq 6$, $2 \leq x \leq 4$ #

**5** The functions f and g are defined by

$$f(x) = e^{2x} - 4, \quad x \in \mathbb{R},$$
$$g(x) = x + 2, \quad x \in \mathbb{R}.$$

**(i)** Find $f^{-1}(x)$ and state its domain. **[3]**

**(ii)** Find the exact solution of $fg(x) = 5$, giving your answer in its simplest form. **[3]**

5 (i) let $y = e^{2x} - 4$

$4 + y = e^{2x}$

$x = \frac{1}{2}\ln|4+y|$

$= \frac{1}{2}\ln(4+y) \quad \because y > -4$

$f^{-1}(x) = \frac{1}{2}\ln(x+4), \quad x > -4$

$D_{f^{-1}} : (-4, \infty)$

5 (ii) $fg(x) = 5$

$g(x) = f^{-1}(5) = \frac{1}{2}\ln(5+4) = \ln 3$

$\therefore x + 2 = \ln 3$

$x = \ln 3 - 2$

| | | |
|---|---|---|
| 6 | (i) By writing $\frac{1}{4r^2-1}$ in partial fractions, find an expression for $\sum_{r=1}^{n} \frac{1}{4r^2-1}$. | [4] |
| | (ii) Hence find the exact value of $\sum_{r=11}^{\infty} \frac{1}{4r^2-1}$. | [2] |

6 (i) $\frac{1}{4r^2-1} = \frac{1}{(2r+1)(2r-1)} = \frac{A}{2r+1} + \frac{B}{2r-1}$

sub in $r = \frac{1}{2}$ , $B = \frac{1}{2}$

$r = -\frac{1}{2}$ , $A = -\frac{1}{2}$

Then $\frac{1}{4r^2-1} = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{2r-1} - \frac{1}{2r+1}$$

$$= \frac{1}{2}\left\{ \begin{array}{l} \frac{1}{1} - \cancel{\frac{1}{3}} \\ \cancel{\frac{1}{3}} - \frac{1}{5} \\ \vdots \\ \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1} \end{array} \right. = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \#$$

6 (ii) $\sum_{r=11}^{\infty} \frac{1}{4r^2-1} = \sum_{r=1}^{\infty} \frac{1}{4r^2-1} - \sum_{r=1}^{10} \frac{1}{4r^2-1}$

$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{21}\right)$

$= \frac{1}{42}$ #

### Suggested Answers

| | |
|---|---|
| **7** A curve $C$ has equation $y = xe^{-x}$. | |
| **(i)** Find the equations of the tangents to $C$ at the points where $x = 1$ and $x = -1$. | **[6]** |
| **(ii)** Find the acute angle between these tangents. | **[2]** |

7 (i) $y = xe^{-x}$

$\frac{dy}{dx} = e^{-x} + x(-e^{-x})$

$\left.\frac{dy}{dx}\right|_{x=1} = e^{-1} - e^{-1} = 0$

$\left.\frac{dy}{dx}\right|_{x=-1} = e^{1} + e^{1} = 2e$

Eq$^{n}$ of tangent at $(1, e^{-1})$

$y = e^{-1}$ $\quad \because \frac{dy}{dx} = 0$

Eq$^{n}$ of tangent at $(-1, -e)$

$y + e = 2e(x+1)$

$y = 2ex + e$ $\quad$ #

(ii) $m = \tan\theta$

so $2e = \tan\theta$

$\theta = 79.57°$

$\approx 79.6°$ //

**8 (a)** An arithmetic series has first term $a$ and common difference $2a$, where $a \neq 0$. A geometric series has first term $a$ and common ratio 2. The $k$th term of the geometric series is equal to the sum of the first 64 terms of the arithmetic series. Find the value of $k$. [3]

**(b)** A geometric series has first term $f$ and common ratio $r$, where $f,\ r \in \mathbf{R}$ and $f \neq 0$. The sum of the first four terms of the series is 0. Find the possible values of $f$ and $r$. Find also, in terms of $f$, the possible values of the sum of the first $n$ terms of the series. [4]

**(c)** The first term of an arithmetic series is negative. The sum of the first four terms of the series is 14 and the product of the first four terms of the series is 0. Find the 11th term of the series. [4]

8 (i) $T_k = a(2^{k-1})$

$$S_{64} = \frac{64}{2}\left(2a + (63)(2a)\right)$$

$$= 4096a$$

$$\Rightarrow 4096a = a2^{k-1}$$

$$12 = k-1$$

$$k = 13 \quad \#$$

8b) $$S_4 = \frac{f(r^4-1)}{r-1} = 0$$

$$\Rightarrow \frac{f(r^2-1)(r^2+1)}{r-1} = 0$$

$$\frac{f(r-1)(r+1)(r^2+1)}{r-1} = 0$$

$$f(r+1)(r^2+1) = 0$$

$f = 0$ (rej) $\quad r = -1 \quad r^2+1 = 0$ (rej)

$\because f \neq 0$

so $f \in \mathbb{R},\ r = -1$ #

$$S_n = \frac{f((-1)^n - 1)}{-1-1} = -\frac{f}{2}((-1)^n - 1) = \begin{cases} 0 & \text{when } n \text{ is even} \\ f & \text{when } n \text{ is odd} \end{cases}$$

#

8c) $a < 0$

$S_4 = \frac{4}{2}(2a + (3)(d)) = 14$

$2a + 3d = 7$

$(a)(a+d)(a+2d)(a+3d) = 0$

So $a = 0$ (rej) $\quad a = -d \quad a = -2d$ or $a = -3d$

$a = -d \Rightarrow -2d + 3d = 7$

$d = 7$

$a = -2d \Rightarrow -4d + 3d = 7$

$d = -7$ (rej) $\because$ a will be positive

$a = -3d \Rightarrow -6d + 3d = 7$

$d = -7/3$ (rej) $\because$ a will be positive

$T_{11} = -7 + (10)(7) = 63$ #

| | |
|---|---|
| **9** **(i)** The complex number $w$ can be expressed as $\cos\theta + \mathrm{i}\sin\theta$.<br>**(a)** Show that $w + \frac{1}{w}$ is a real number. | [2] |
| **(b)** Show that $\frac{w-1}{w+1}$ can be expressed as $k\tan\frac{1}{2}\theta$, where $k$ is a complex number to be found. | [4] |
| **(ii)** The complex number $z$ has modulus 1. Find the modulus of the complex number $\frac{z-3\mathrm{i}}{1+3\mathrm{i}z}$. | [5] |

9 (i)(a) $w = \cos\theta + i\sin\theta$

$= e^{i\theta}$

$e^{i\theta} + \frac{1}{e^{i\theta}} = e^{i\theta} + e^{-i\theta}$

$= \cos\theta + i\sin\theta + \cos(\theta) + i(\sin(-\theta))$

$= 2\cos\theta \in \mathbb{R}$ #

9(i)(b) $\frac{w-1}{w+1} = \frac{e^{i\theta}-1}{e^{i\theta}+1} = \frac{e^{i\frac{1}{2}\theta}\left(e^{i\frac{1}{2}\theta} - e^{-\frac{1}{2}\theta i}\right)}{e^{i\frac{1}{2}\theta}\left(e^{\frac{1}{2}\theta i} + e^{-\frac{1}{2}\theta i}\right)}$

$= \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} - \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(-\frac{\theta}{2}\right)\right)}{2\cos\frac{\theta}{2}}$

$= \frac{2i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = 2i\tan\frac{\theta}{2}$

$\therefore k = 2i$ #

9(ii) $|z|=1$ then $z = e^{i\theta}$

$$\left|\frac{z-3i}{1+3iz}\right| = \left|\frac{z\left(1-\frac{3i}{z}\right)}{1+3iz}\right| = |z|\left|\frac{1-3iz^*}{1+3iz}\right|$$

$$= |z|\left|\frac{1^* + 3(-i)z^*}{1+3iz}\right|$$

$$= |z|\left|\frac{1^* + (3iz)^*}{1+3iz}\right|$$

$$= |z|\left|\frac{(1+3iz)^*}{1+3iz}\right|$$

$$= |z| = 1 \ \#$$

**10** A curve $C$ has parametric equations

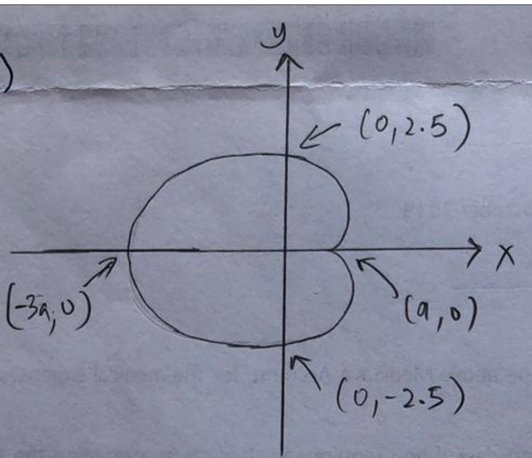
$$x = a(2\cos\theta - \cos 2\theta),$$
$$y = a(2\sin\theta - \sin 2\theta),$$

for $0 \leqslant \theta \leqslant 2\pi$.

**(i)** Sketch $C$ and state the Cartesian equation of its line of symmetry. **[2]**

**(ii)** Find the values of $\theta$ at the points where $C$ meets the $x$-axis. **[2]**

10)(i)




line of symmetry $y=0$

(ii) $\theta = 0$, $\theta = \pi$, $\theta = 2\pi$ from G.C.

| | |
|---|---|
| **(iii)** Show that the area enclosed by the $x$-axis, and the part of $C$ above the $x$-axis, is given by $$\int_{\theta_1}^{\theta_2} a^2(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta,$$ where $\theta_1$ and $\theta_2$ should be stated. | [3] |
| **(iv)** Hence find, in terms of $a$, the exact total area enclosed by $C$. | [5] |

(iii) $\int_{-3a}^{a} y\,dx$ $\qquad \frac{dx}{d\theta} = a(-2\sin\theta + 2\sin 2\theta)$

$= \int_{\pi}^{0} a(2\sin\theta - \sin 2\theta)(a(-2\sin\theta + 2\sin 2\theta))\,d\theta$

$= \int_{\pi}^{0} a^2(-4\sin^2\theta + 4\sin\theta\sin 2\theta + 2\sin\theta\sin 2\theta - 2\sin^2 2\theta)\,d\theta$

$= \int_{\pi}^{0} a^2(-4\sin^2\theta + 6\sin\theta\sin 2\theta - 2\sin^2 2\theta)\,d\theta$

$= \int_{0}^{\pi} a^2(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta$ # shown

$\theta_1 = 0 \qquad \theta_2 = \pi$

10 (iv) $\int_{0}^{\pi} a^2(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta$

$= 3\pi a^2$ // from G.C.

**11** Scientists are investigating how the temperature of water changes in various environments.

**(i)** The scientists begin by investigating how hot water cools.

The water is heated in a container and then placed in a room which is kept at a constant temperature of 16 °C. The temperature of the water $t$ minutes after it is placed in the room is $\theta$ °C. This temperature decreases at a rate proportional to the difference between the temperature of the water and the temperature of the room. The temperature of the water falls from a value of 80 °C to 32 °C in the first 30 minutes.

**(a)** Write down a differential equation for this situation. Solve this differential equation to get $\theta$ as an exact function of $t$. [6]

11 (i) Let $\theta$ be the temp of water at time $t$.

$$\frac{d\theta}{dt} = -k(\theta - 16)$$

$$\int \frac{1}{\theta - 16}\, d\theta = \int -k\, dt$$

$$\ln|\theta - 16| = -kt + c$$

$$\theta - 16 = Ae^{-kt} \quad \text{where } A = \pm e^{c}$$

$$\theta = 16 + Ae^{-kt}$$

$t = 0$, $\theta = 80°$

$$80 = 16 + A$$

$$\Rightarrow A = 64$$

$$\theta = 16 + 64e^{-kt}$$

$t = 30$, $\theta = 32$

$$\frac{1}{4} = e^{-k(30)}$$

$$\ln\frac{1}{4} = -30k$$

$$\ln 4 = 30k$$

$$k = \frac{1}{30}\ln 4$$

$$\therefore \theta = 16 + 64e^{-\frac{t}{30}\ln 4}$$

$$= 16 + 64(4)^{-t/30}$$

#

**(b)** Find the temperature of the water 45 minutes after it is placed in the room. [1]

**11 [Continued]**

**(ii)** The scientists then model the thickness of ice on a pond.

In winter the surface of the water in the pond freezes. Once the thickness of the ice reaches 3 cm, it is safe to skate on the ice. The thickness of the ice is $T$ cm, $t$ minutes after the water starts to freeze. The freezing of the water is modelled by a differential equation in which the rate of change of the thickness of the ice is inversely proportional to its thickness. It is given that $T = 0$ when $t = 0$. After 60 minutes, the ice is 1 cm thick.

Find the time from when freezing commences until the ice is first safe to skate on. [6]

11 (i) b) $\theta = 16 + 64(4)^{-t/30}$

$\theta = 16 + 64(4)^{-1.5}$

$= 24°C$ #

11 (ii) $\frac{dT}{dt} = \frac{k}{T}$

$\int T \, dT = \int k \, dt$

$\frac{1}{2}T^2 = kt + c$

$t = 0$, $T = 0$ then $c = 0$

$t = 60$, $T = 1$

$\frac{1}{2} = k(60)$

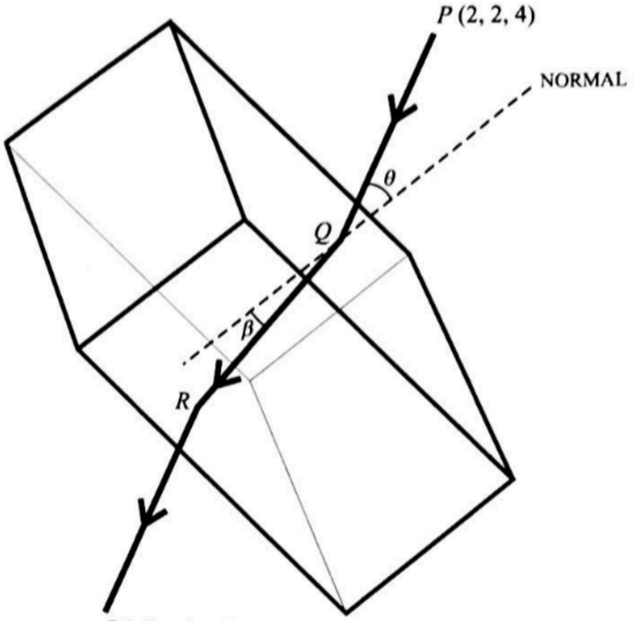
$k = \frac{1}{120}$

$\Rightarrow \frac{1}{2}T^2 = \frac{1}{120}t$

$T = 3$ $\quad \frac{1}{2}(9) = \frac{1}{120}t$

$t = 540$ mins #

**12**




A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in direction $\begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$ from a light source at the point $P$ with coordinates $(2, 2, 4)$. The prism is placed so that the ray of light passes through the prism, entering at the point $Q$ and emerging at the point $R$ and is picked up by a sensor at point $S$ with coordinates $(-5, -6, -7)$. The acute angle between $PQ$ and the normal to the top of the prism at $Q$ is $\theta$ and the acute angle between $QR$ and the same normal is $\beta$ (see diagram).

It is given that the top of the prism is a part of the plane $x + y + z = 1$, and that the base of the prism is a part of the plane $x + y + z = -9$. It is also given that the ray of light along $PQ$ is parallel to the ray of light along $RS$ so that $P$, $Q$, $R$ and $S$ lie in the same plane.

**(i)** Find the exact coordinates of $Q$ and $R$. [5]

**(ii)** Find the values of $\cos\theta$ and $\cos\beta$. [3]

**(iii)** Find the thickness of the prism measured in the direction of the normal at $Q$. [3]

Snell's law states that $\sin\theta = k\sin\beta$, where $k$ is a constant called the refractive index.

**(iv)** Find $k$ for the material of this prism. [1]

**(v)** What can be said about the value of $k$ for a material for which $\beta > \theta$? [1]

12 (i) $\text{line}_{PQ}: \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $\alpha \in \mathbb{R}$

$\text{Plane}_{Top}: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$

$$\begin{pmatrix} 2+2\alpha \\ 2+3\alpha \\ 4+6\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$8 + 11\alpha = 1$$

$$\alpha = -7/11$$

$$\therefore \vec{OQ} = \begin{pmatrix} 2 - 14/11 \\ 2 - 21/11 \\ 4 - 42/11 \end{pmatrix} = \begin{pmatrix} 8/11 \\ 1/11 \\ 2/11 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore$ coordinates $Q\left(\frac{8}{11}, \frac{1}{11}, \frac{2}{11}\right)$

$\text{line}_{RS}: \mathbf{r} = \begin{pmatrix} -5 \\ -6 \\ -7 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

$\text{Plane}_{bottom}: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -9$

$$\begin{pmatrix} -5+2\alpha \\ -6+3\alpha \\ -7+6\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -9$$

$$-18 + 11\alpha = -9$$

$$\alpha = \frac{9}{11}$$

$$\vec{OR} = \begin{pmatrix} -5 + 18/11 \\ -6 + 27/11 \\ -7 + 54/11 \end{pmatrix} = \begin{pmatrix} -37/11 \\ -39/11 \\ -23/11 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} 37 \\ 39 \\ 23 \end{pmatrix}$$

$\therefore$ Coordinates $R\left(-\frac{37}{11}, -\frac{39}{11}, -\frac{23}{11}\right)$ #

### Suggested Answers

12(ii) $\cos\theta = \dfrac{\left|\begin{pmatrix}2\\3\\6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\3\\6\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|} = \dfrac{11\sqrt{3}}{21}$

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

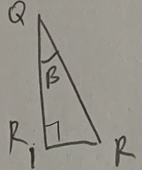
$= \frac{1}{11}\begin{pmatrix}-8\\-1\\-2\end{pmatrix} + \frac{1}{11}\begin{pmatrix}-37\\-39\\-23\end{pmatrix}$

$= -\frac{1}{11}\begin{pmatrix}45\\40\\25\end{pmatrix}$

$= -\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}$

$\cos\beta = \dfrac{\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|} = \dfrac{22}{\sqrt{510}} = \dfrac{11\sqrt{510}}{255}$ #

12(iii)



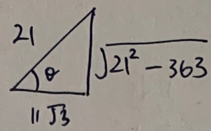

$\cos\beta = \dfrac{|\overrightarrow{QR_1}|}{|\overrightarrow{QR}|} = \dfrac{|\overrightarrow{QR_1}|}{\frac{5}{11}\sqrt{170}}$

$\therefore |\overrightarrow{QR_1}| = \dfrac{\left(\frac{5}{11}\sqrt{170}\right)\left(11\sqrt{510}\right)}{255}$

$= \dfrac{5(170)\sqrt{3}}{255}$

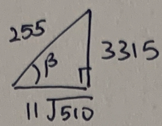
$= \dfrac{10\sqrt{3}}{3}$ //

(iv) $\cos\theta = \dfrac{11\sqrt{3}}{21}$




$\sin\theta = \dfrac{\sqrt{78}}{21}$

$\cos\beta = \dfrac{11\sqrt{510}}{255}$




$\sin\beta = \dfrac{\sqrt{3315}}{255}$

$$k = \frac{\sin\theta}{\sin\beta} = \frac{\sqrt{78}}{21} \times \frac{255}{\sqrt{3315}} = \frac{85}{7}\left(\frac{\sqrt{2}}{\sqrt{85}}\right) = \frac{\sqrt{170}}{7} \approx 1.86$$

(v) $\theta = \cos^{-1}\dfrac{11\sqrt{3}}{21}$ $\qquad$ $\beta = \cos^{-1}\dfrac{11\sqrt{510}}{255}$

$\approx 24.9^\circ$ $\qquad$ $\approx 13.0^\circ$

if $\beta > \theta$, then $k < 1$ #