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MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME	Lim

CENTRE NUMBER	S					INDEX NUMBER	

MATHEMATICS

9758/01

Paper 1

October/November 2019

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Solution served as a suggestion only

#1 Math Tuition

1 The function f is defined by $f(z) = az^3 + bz^2 + cz + d$, where a, b, c and d are real numbers. Given that $2 + i$ and -3 are roots of $f(z) = 0$, find b, c and d in terms of a. [4]

$\because$ all coefficients are real, then $2-i$ is also a root

$$f(z) = (z-(2+i))(z-(2-i))(z+3)(a)$$

$$= ((z-2)^2+1)(z+3)(a)$$

$$= (z^3 - z^2 - 7z + 15)(a)$$

$\therefore b = -a$, $c = -7a$ $d = 15a$

2 The curve C has equation $y = x^3 + x - 1$.

(i) C crosses the x-axis at the point with coordinates $(a, 0)$. Find the value of a correct to 3 decimal places. [1]

$a = 0.682$ G.C.

(ii) You are given that $b > a$.

The region P is bounded by C, the x-axis and the lines $x = -1$ and $x = 0$. The region Q is bounded by C, the line $x = b$ and the part of the x-axis between $x = a$ and $x = b$. Given that the area of Q is 2 times the area of P, find the value of b correct to 3 decimal places. [4]

$$\int_{-1}^{0} \left| x^3 + x - 1 \right| dx = 1.75 \quad (\text{G.C})$$

$$\int_a^b x^3 + x - 1 \, dx = 3.5$$

$$\left[\frac{1}{4}x^4 + \frac{1}{2}x^2 - x \right]_a^b = 3.5$$

$\because a = 0.682$

$$\left(\frac{1}{4}b^4 + \frac{1}{2}b^2 - b \right) - (-0.3953) = 3.5$$

From G.C. $b = 1.892$ $\because b > a$

3 A function is defined as $f(x) = 2x^3 - 6x^2 + 6x - 12$.

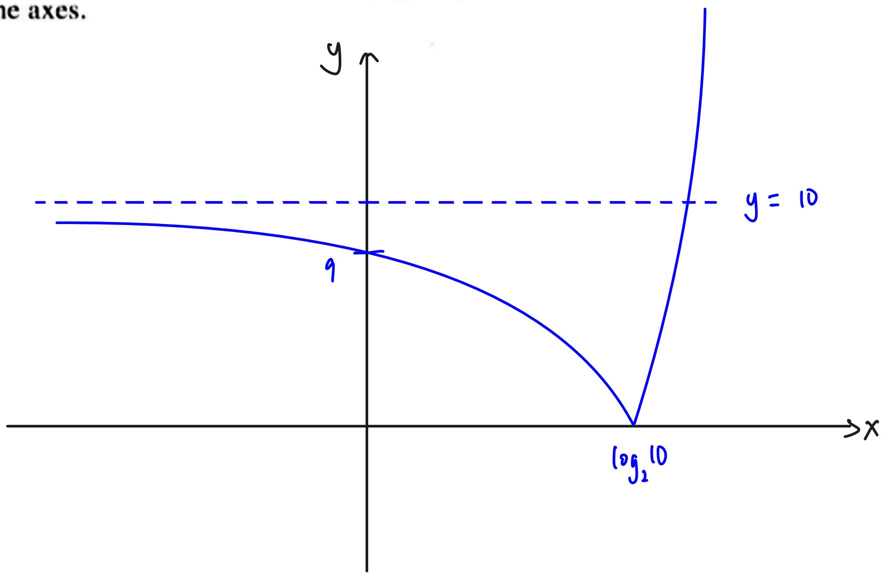
(i) Show that $f(x)$ can be written in the form $p\{(x+q)^3 + r\}$, where p, q and r are constants to be found. [2]

$f(x) = 2\left((x-1)^3 - 5\right)$

(ii) Hence, or otherwise, describe a sequence of transformations that transform the graph of $y = x^3$ onto the graph of $y = f(x)$. [3]

1. Translation 1 unit in positive x-axis direction
2. Translation 5 units in negative y-axis direction
3. Scaling 2 units parallel to the y-axis.

4 **(i)** Sketch the graph of $y = |2^x - 10|$, giving the exact values of any points where the curve meets the axes. [3]



Solution served as a suggestion only

(ii) Without using a calculator, and showing all your working, find the exact interval, or intervals, for which $|2^x - 10| \leqslant 6$. Give your answer in its simplest form. [3]

$10 - 2^x = 6$ $\qquad$ $2^x - 10 = 6$

$x = 2$ $\qquad$ $x = 4$

$2 \leq x \leq 4$ #

5 The functions f and g are defined by

$$f(x) = e^{2x} - 4, \quad x \in \mathbb{R},$$
$$g(x) = x + 2, \quad x \in \mathbb{R}.$$

(i) Find $f^{-1}(x)$ and state its domain. [3]

let $y = e^{2x} - 4$

$x = \frac{1}{2}\ln(4+y) \quad \because y > -4$

$\therefore f^{-1}(x) = \frac{1}{2}\ln(4+x), \quad x > -4$

$D_{f^{-1}} : (-4, \infty)$

DO NOT WRITE IN THIS MARGIN

Solution served as a suggestion only

(ii) Find the exact solution of $\mathrm{fg}(x) = 5$, giving your answer in its simplest form. [3]

$\mathrm{fg}(x) = 5$

$g(x) = f^{-1}(5)$

$= \frac{1}{2}\ln(5+4)$

$\therefore x+2 = \ln 3$

$x = -2 + \ln 3$ #

6 (i) By writing $\dfrac{1}{4r^2-1}$ in partial fractions, find an expression for $\displaystyle\sum_{r=1}^{n}\dfrac{1}{4r^2-1}$. [4]

$$\sum_{r=1}^{n}\frac{1}{4r^2-1} = \sum_{r=1}^{n}\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$$

$$= \frac{1}{2}\left\{\begin{array}{l}\frac{1}{1}-\cancel{\frac{1}{3}} \\ \cancel{\frac{1}{3}}-\cancel{\frac{1}{5}} \\ \vdots \\ \cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}\end{array}\right. = \frac{1}{2}\left(1-\frac{1}{2n+1}\right) \#$$

(ii) Hence find the exact value of $\sum_{r=11}^{\infty} \frac{1}{4r^2 - 1}$. [2]

$$\sum_{r=11}^{\infty} \frac{1}{4r^2-1} = \sum_{r=1}^{\infty} \frac{1}{4r^2-1} - \sum_{r=1}^{10} \frac{1}{4r^2-1}$$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{21}\right) = \frac{1}{42} \ \#$$

7 A curve C has equation $y = xe^{-x}$.

(i) Find the equations of the tangents to C at the points where $x = 1$ and $x = -1$. [6]

$y = xe^{-x}$

$\frac{dy}{dx} = e^{-x} - xe^{-x}$

$\left.\frac{dy}{dx}\right|_{x=1} = e^{-1} - e^{-1} = 0$ $\left.\frac{dy}{dx}\right|_{x=-1} = e^{1} + e^{1} = 2e$

Eq$^{\underline{n}}$ of tangent at $(1, e^{-1})$

$y = e^{-1}$

Eq$^{\underline{n}}$ of tangent at $(-1, -e)$

$y + e = 2e(x+1)$

$y = 2ex + e$ #

(ii) Find the acute angle between these tangents. **[2]**

$\tan\theta = 2e$

$\theta = 79.6°$

Solution served as a suggestion only

8 **(a)** An arithmetic series has first term a and common difference $2a$, where $a \neq 0$. A geometric series has first term a and common ratio 2. The kth term of the geometric series is equal to the sum of the first 64 terms of the arithmetic series. Find the value of k. [3]

$$T_k = a\left(2^{k-1}\right) \qquad S_{64} = \frac{64}{2}\left(2a + 63(2a)\right)$$

$$= 4096a$$

$$4096a = a\left(2^{k-1}\right)$$

$$k = 13 \quad \#$$

(b) A geometric series has first term f and common ratio r, where $f, r \in \mathbf{R}$ and $f \neq 0$. The sum of the first four terms of the series is 0. Find the possible values of f and r. Find also, in terms of f, the possible values of the sum of the first n terms of the series. [4]

$$S_4 = \frac{f(r^4-1)}{r-1} = 0$$

$$\frac{f(r^2-1)(r^2+1)}{r-1} = 0$$

$\Rightarrow f=0$ (rej $\because f\neq 0$) $\quad r=-1 \quad r^2+1=0$ (rej) $\quad r=1$ (rej $\because$ sum will be $4f$)

$$S_n = \frac{f((-1)^n-1)}{-1-1} = -\frac{f}{2}\left((-1)^n-1\right) = \begin{cases} f & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

(c) The first term of an arithmetic series is negative. The sum of the first four terms of the series is 14 and the product of the first four terms of the series is 0. Find the 11th term of the series. [4]

$S_4 = \frac{4}{2}(2a + 3d) = 14$

$2a + 3d = 7$

$a(a+d)(a+2d)(a+3d) = 0$

so $a = 0$ (rej) $\quad a = -d \quad a = -2d \quad a = -3d$

if $a = -d \Rightarrow -2d + 3d = 7$
$d = 7$

if $a = -2d \Rightarrow d = -7$ (rej) $\because a > 0$

if $a = -3d \Rightarrow d = -7/3$ (rej) $\because a > 0$

$\therefore T_{11} = -7 + 10(7) = 63$ *

9 **(i)** The complex number w can be expressed as $\cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $w + \dfrac{1}{w}$ is a real number. [2]

$$w + \frac{1}{w} = e^{i\theta} + e^{-i\theta}$$

$$= \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)$$

$$= 2\cos\theta \in \mathbb{R}$$ ✱ shown

(b) Show that $\dfrac{w-1}{w+1}$ can be expressed as $k\tan\frac{1}{2}\theta$, where k is a complex number to be found. [4]

$$\frac{W-1}{W+1} = \frac{e^{i\theta}-1}{e^{i\theta}+1} = \frac{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right)}$$

$$= \frac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = i\tan\frac{\theta}{2}$$

$$k = i$$

(ii) The complex number z has modulus 1. Find the modulus of the complex number $\dfrac{z-3\mathrm{i}}{1+3\mathrm{i}z}$. [5]

$$\left|\frac{z-3i}{1+3iz}\right| = \left|\frac{z\left(1+\frac{3i}{z}\right)}{1+3iz}\right|$$

$$= |z|\left|\frac{1^* + 3(-i)(z^*)}{1+3iz}\right| \qquad \because\ z^* = z$$

$$= \left|\frac{1^* + (3iz)^*}{1+3iz}\right|$$

$$= \left|\frac{(1+3iz)^*}{1+3iz}\right| = 1$$

Solution served as a suggestion only

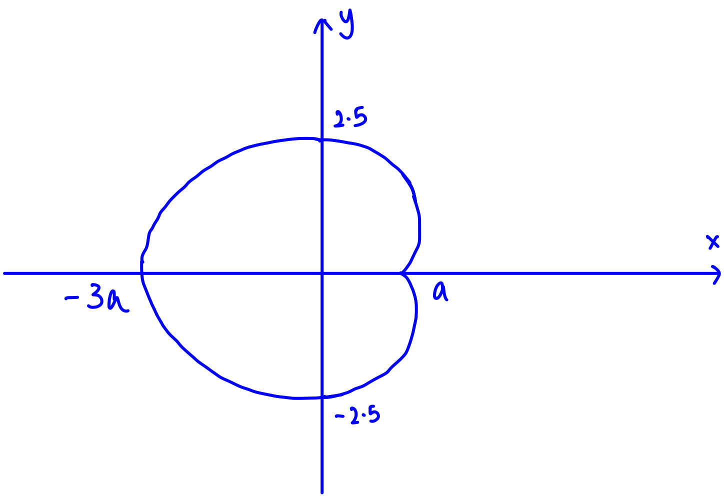
#1 Math Tuition

10 A curve C has parametric equations

$$x = a(2\cos\theta - \cos 2\theta),$$
$$y = a(2\sin\theta - \sin 2\theta),$$

for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C and state the Cartesian equation of its line of symmetry. [2]



line of symmetry : $y = 0$

(ii) Find the values of θ at the points where C meets the x-axis. [2]

$\theta = 0, \pi, 2\pi$

(iii) Show that the area enclosed by the x-axis, and the part of C above the x-axis, is given by

$$\int_{\theta_1}^{\theta_2} a^2(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta,$$

where θ_1 and θ_2 should be stated. [3]

$$\frac{dx}{d\theta} = a(-2\sin\theta + 2\sin 2\theta)$$

$$\int_{-3a}^{a} y\,dx$$

$$= \int_{\pi}^{0} a(2\sin\theta)\left(a(-2\sin\theta + 2\sin 2\theta)\right)d\theta$$

$$= \int_{\pi}^{0} a^2(-4\sin^2\theta + 4\sin\theta\sin 2\theta + 2\sin\theta\sin 2\theta - 2\sin^2 2\theta)\,d\theta$$

$$= \int_{0}^{\pi} a^2(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta$$

$\theta_1 = 0 \quad \theta_2 = \pi$

(iv) Hence find, in terms of a, the exact total area enclosed by C. [5]

$$\int_{0}^{\pi} a^2(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta)\,d\theta$$

$$= 3\pi a^2$$

$\therefore$ Total area $6\pi a^2$ //

11 Scientists are investigating how the temperature of water changes in various environments.

(i) The scientists begin by investigating how hot water cools.

The water is heated in a container and then placed in a room which is kept at a constant temperature of 16 °C. The temperature of the water t minutes after it is placed in the room is θ °C. This temperature decreases at a rate proportional to the difference between the temperature of the water and the temperature of the room. The temperature of the water falls from a value of 80 °C to 32 °C in the first 30 minutes.

(a) Write down a differential equation for this situation. Solve this differential equation to get θ as an exact function of t. [6]

Let θ be the temp of water at time t.

$$\frac{d\theta}{dt} = -k(\theta - 16)$$

$$\int \frac{1}{\theta - 16}\, d\theta = \int -k\, dt$$

$$\ln|\theta - 16| = -kt + c$$

$$\theta = 16 + Ae^{-kt} \quad \text{where } A = \pm e^{c}$$

$t = 0$, $\theta = 80°$ $\Rightarrow$ $A = 64$

$$\theta = 16 + 64e^{-kt}$$

$t = 30$, $\theta = 32$

$$\frac{1}{4} = e^{-30k} \quad \Rightarrow \quad k = \frac{1}{30}\ln 4$$

$$\therefore \theta = 16 + 64(4)^{-\frac{t}{30}}$$

Solution served as a suggestion only

(b) Find the temperature of the water 45 minutes after it is placed in the room. [1]

$\theta = 16 + 64(4)^{-1.5}$

$= 24°C$

11 [Continued]

(ii) The scientists then model the thickness of ice on a pond.

In winter the surface of the water in the pond freezes. Once the thickness of the ice reaches 3 cm, it is safe to skate on the ice. The thickness of the ice is T cm, t minutes after the water starts to freeze. The freezing of the water is modelled by a differential equation in which the rate of change of the thickness of the ice is inversely proportional to its thickness. It is given that $T = 0$ when $t = 0$. After 60 minutes, the ice is 1 cm thick.

Find the time from when freezing commences until the ice is first safe to skate on. [6]

$$\frac{dT}{dt} = \frac{k}{T}$$

$$\int T \, dT = \int k \, dt$$

$$\frac{1}{2}T^2 = kt + c$$

$t = 0$, $T = 0$ then $c = 0$

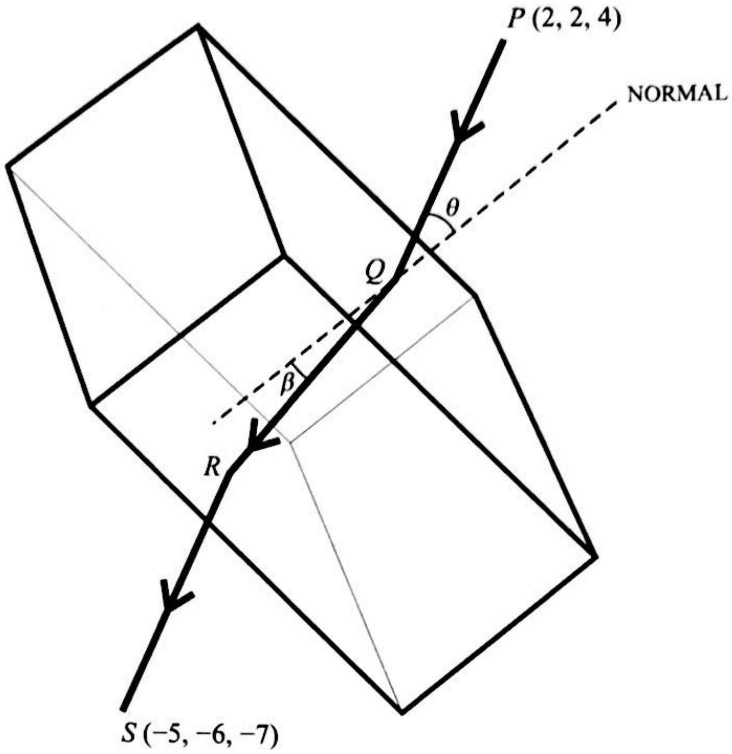
$t = 60$ $T = 1$

$\frac{1}{2} = 60k$ $\therefore k = \frac{1}{120}$

$$\frac{1}{2}T^2 = \frac{1}{120}t$$

when $T = 3$ $t = 540$ mins #

12



A ray of light passes from air into a material made into a rectangular prism. The ray of light is sent in direction $\begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}$ from a light source at the point P with coordinates $(2, 2, 4)$. The prism is placed so that the ray of light passes through the prism, entering at the point Q and emerging at the point R and is picked up by a sensor at point S with coordinates $(-5, -6, -7)$. The acute angle between PQ and the normal to the top of the prism at Q is θ and the acute angle between QR and the same normal is β (see diagram).

It is given that the top of the prism is a part of the plane $x + y + z = 1$, and that the base of the prism is a part of the plane $x + y + z = -9$. It is also given that the ray of light along PQ is parallel to the ray of light along RS so that P, Q, R and S lie in the same plane.

(i) Find the exact coordinates of Q and R. **[5]**

$l_{PQ}: \ r = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \quad \alpha \in \mathbb{R}$

$\text{Plane}_{top}: \ r \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$

$\begin{pmatrix} 2+2\alpha \\ 2+3\alpha \\ 4+6\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \quad \Rightarrow \alpha = -7/11$

$\therefore \ \overrightarrow{OQ} = \frac{1}{11}\begin{pmatrix} 8 \\ 1 \\ 2 \end{pmatrix} \qquad Q\left(\frac{8}{11}, \frac{1}{11}, \frac{2}{11}\right)$

$l_{RS}: \ r = \begin{pmatrix} -5 \\ -6 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

$\text{Plane}_{btm}: \ r \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -9$

$\begin{pmatrix} -5+2\lambda \\ -6+3\lambda \\ -7+6\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -9 \quad \Rightarrow \lambda = \frac{9}{11}$

$\overrightarrow{OR} = -\frac{1}{11}\begin{pmatrix} 37 \\ 39 \\ 23 \end{pmatrix} \qquad R\left(-\frac{37}{11}, -\frac{39}{11}, -\frac{23}{11}\right)$

Solution served as a suggestion only

12 [Continued]

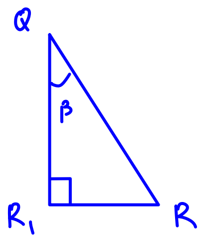
(ii) Find the values of $\cos\theta$ and $\cos\beta$. [3]

$$\cos\theta = \frac{\left|\begin{pmatrix}2\\3\\6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\left|\begin{matrix}2\\3\\6\end{matrix}\right|\left|\begin{matrix}1\\1\\1\end{matrix}\right|} = \frac{11\sqrt{3}}{21}$$

$$\overrightarrow{QR} = \frac{1}{11}\begin{pmatrix}-8\\-1\\-2\end{pmatrix} + \frac{1}{11}\begin{pmatrix}-37\\-39\\-23\end{pmatrix} = -\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}$$

$$\cos\beta = \frac{\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\left|\begin{matrix}9\\8\\5\end{matrix}\right|\left|\begin{matrix}1\\1\\1\end{matrix}\right|} = \frac{11\sqrt{510}}{255} \quad \#$$

(iii) Find the thickness of the prism measured in the direction of the normal at Q. [3]



$$\cos\beta = \frac{|\overrightarrow{QR_1}|}{|\overrightarrow{QR}|} = \frac{|\overrightarrow{QR_1}|}{\frac{5}{11}\sqrt{170}}$$

$$\therefore |\overrightarrow{QR_1}| = \frac{\left(\frac{5}{11}\sqrt{170}\right)\left(11\sqrt{510}\right)}{255}$$

$$= \frac{10\sqrt{3}}{3}$$

Solution served as a suggestion only

Snell's law states that $\sin\theta = k\sin\beta$, where k is a constant called the refractive index.

(iv) Find k for the material of this prism. [1]

$\cos\theta = \frac{11\sqrt{3}}{21} \Rightarrow \sin\theta = \frac{\sqrt{78}}{21}$

$\cos\beta = \frac{11\sqrt{510}}{255} \Rightarrow \sin\beta = \frac{\sqrt{3315}}{255}$

$k = \frac{\sin\theta}{\sin\beta} = \frac{\sqrt{78}}{21} \times \frac{255}{\sqrt{3315}} = \frac{\sqrt{170}}{7} \approx 1.86$

(v) What can be said about the value of k for a material for which $\beta > \theta$? [1]

$\theta = \cos^{-1}\frac{11\sqrt{3}}{21}$ $\quad\quad$ $\beta = \cos^{-1}\frac{11\sqrt{510}}{255}$

$\approx 24.9^\circ$ $\quad\quad$ $\approx 13.0^\circ$

If $\beta > \theta$ then $0 < k < 1$